Navigating the Infinite Web of Pitch Space

Kyle Gann

Given any two pitches, a third pitch is almost always implied.

For example, say I have a tone vibrating at 550 cycles per second (C#) and another at 660 cps (E). Both pitches are harmonics of a third pitch that vibrates at 110 cps (A). (That is, 110 is the greatest common denominator of 550 and 660.)

The exception to this opening statement is the case in which one pitch is an overtone of another. If I have a tone at 440 cps (A), and another at 880 (A an octave higher), no third pitch is implied. This simple consonance does not require (or suggest) <u>mediation</u> by a third pitch.

Let us consider pitches more distantly related. Suppose I have two pitches vibrating at 400 and 270 cps respectively. Both are harmonics (the 40th and 27th, respectively) of another pitch vibrating at 10 cps. But 10 cps is too slow a vibration to register as sound. By bringing 10 cps up several octaves

10 x 2 = 20
20 x 2 = 40
40 x 2 = 80
80 x 2 = 160

we arrive at a pitch more simply related to both 400 and 270 than either is to the other. 400/160 (5/2) and 270/160 (27/16) are simpler fractions than 400/270 (40/27). They can get by with smaller numbers, and are therefore more consonant. Consonant is often taken to mean sweet-sounding or pleasant, but it really means <u>intelligible</u>. By definition, the smaller the numbers in a pitch ratio, the more consonant it is.

One could say that pitches represent a family situation. Take two people. You check their backgrounds and find out that they are fourth cousins to each other. This implies a common ancestor. If that common ancestor is so far back in history as to be no longer living, then find the parent of one of the persons, and that parent may be more closely related to each of them than they are to each other. We say these days that, given any two people, there are no more than six degrees of separation between them: sufficient common denominators can be found to link them together and show their relation. Similarly, any two pitches are related in some way, perhaps more closely, perhaps less closely - but the relationship between them can always be found. The more distant it is, the more intermediary pitches are implied.

More accurately, since we are dealing with arithmetic and not human relationships, pitches exist in a web situation. Points in a web are all connected to each other. The further apart two pitches are in the web, the more different pitches are touched on in trying to connect them. Why do we try to connect them? To make their coexistence <u>intelligible</u>.

Let's take another approach to our example. Played together, pitches at 400 and 270 cps form an interval denoted by the fraction or ratio 40/27. The ratio means that if we take 270 cps as 1 (or 1/1), 400 cps is represented by 40/27. This ratio can be factored out:

$$\frac{40}{27} = \frac{2 \times 2 \times 2 \times 5}{3 \times 3 \times 3}$$

Any subset of these numbers will provide us with pitches more closely related to both 1/1 and 40/27 than they are to each other. For instance,

$$\frac{2 \times 2}{3} = \frac{4}{3} \qquad \frac{2 \times 5}{3 \times 3} = \frac{10}{9}$$

$$\frac{5}{3} = \frac{5}{3} \qquad \frac{2 \times 2 \times 2}{3 \times 3} = \frac{8}{9}$$

$$\frac{2 \times 2 \times 2}{3 \times 3 \times 3} = \frac{8}{27} \qquad \frac{5}{3^0} = \frac{5}{1}$$

This gives us six pitches, each more consonant with both 1/1 and 40/27 than those two pitches are with each other. In fact, discounting pitches that are octaves of each other (multiplication or division of a pitch by 2 always produces an octave, another pitch with the same pitch name), these are the only six pitches more consonant with both those pitches than they are with each other. Ignoring further multiplications by 2, we've gone through all the available permutations.[1]

This provides us, then, with a scale whose members make 40/27 and 1/1 intelligible (I have multiplied ratios by 2 or 4 where necessary - that is, transposed them up or down an octave or two - to bring them all into the same octave):

16/9	(2 x 8/9)
5/3	
40/27	
4/3	
5/4	(5/1 x 1/4)
32/27	(4 x 8/27)
10/9	
1/1	

Translated into musical terms, this produces a scale as follows:

Ratio:	Pitch name:	Cents above C:
16/9	Bb-	996.1...
5/3	A	884.4...
40/27	G-	680.4...
4/3	F	498.0...
5/4	E	386.3...
32/27	Eb-	294.1...
10/9	D-	182.4...
1/1	C	0.0

The minuses in the pitch names are taken from a tuning-specific pitch notation developed by Ben Johnston, of which more will be said later. Briefly, they indicate that the pitch has been lowered by 21.5 cents, or a 81/80 ratio (called the syntonic comma).

The solution of this problem is of particular interest, because the interval 40/27 has had a notorious history in European music. It represents the so-called "wolf fifth," the "wolf" interval one finds between D and A in a scale tuned to pure C major. That is, in the scale

C	D	E	F	G	A	B	C
1/1	9/8	5/4	4/3	3/2	5/3	15/8	2/1

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[1] Those who still find this method of describing pitch relationships confusing might find further simplification in an article on my web page at http://home.earthlink.net/~kgann/tuning.html.

the interval D to A is 9/8 to 5/3, and 5/3 ÷ 9/8 = 5/3 x 8/9 = 40/27. In the major scale as given here, the so-called I, IV, and V chords are perfectly in tune, but the ii chord on D is simply unavailable because it sounds woefully out of tune; its fifth is 21.5 cents flat, causing audible beats between the two tones likened to a wolf's howl. In 16th-century Italy, the existence and seeming unavoidability of this interval led to experimentation with harpsichords containing notes with more than 12 pitches per octave. As late as the theoretical writings of Anton Bruckner in the 19th century, the interval between the second and sixth scale degrees is to be treated as a dissonance and resolved as such, not taken as a "normal" perfect fifth of 3/2. To avoid the wolf fifth entails one set of difficulties, with which Europe struggled for centuries; to embrace the wolf fifth, to make it intelligible entails a different set.

Perhaps, ultimately, a more interesting set.

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In 1984 an offhand comment by my composition teacher Ben Johnston sparked in me what would soon become an enduring interest in pure, arithmetically defined tunings. Several years later, I acquired a synthesizer that allowed me to retune each pitch to within 6/10 of a cent of whatever I wanted. By this time I was over 30 years old, and had been composing for two decades in a 12-pitch, equal-spaced scale, the same one most American and European musicians use. Suddenly I found myself faced with an interior infinity of possible pitches. The landscape of pitches had seemed simple and firmly anchored in a base-12 arithmetic. Now I was without moorings, floating free from the gravity of the fixed half-step. To be intuitively creative in such fluid and unfamiliar circumstances was, for me, impossible. Not until 1992 did I manage to write a piece of music that could only be conceived in terms of pure microtonal tunings. The gestation process took eight years, during which my theoretical speculations were virtually continuous, manifested in entire notebooks filled with charts of multidimensionally nested fractions.

The step-by-step logic I outline in the opening paragraphs of this paper represents the logic by which I eventually came to map my new pitch universe. There are certainly other ways to do it - Harry Partch, La Monte Young, and Ben Johnston himself had already come up with different, if sometimes parallel paths.

Briefly, and to oversimplify, Young in his The Well-Tuned Piano had come upon a 12-pitch tuning that provided new intervals on the piano, intervals based on the number 7 rather than on the European number 5. But I was not using piano, for the most practical of reasons (a piano must be tuned and retuned several times before and after a microtonal performance, vastly diminishing the chance of one's getting performed), and so I was not limited to 12 pitches. In his sine-tone installations, Young works with closely-spaced harmonics in the upper reaches of the harmonic series. But common, performance-oriented synthesizers did not provide me with sufficiently exact pitch resolution to climb that high in the harmonic series. Partch derived scales and tonalities from symmetrical overlays of overtone series' and undertone series'; to some extent, Ben followed him in this direction. But first of all, I wasn't entirely convinced of the relevance of an undertone series, and anyway I found by experimentation that, with the limitations of tuning resolution on my synthesizers, undertone chords sounded unusably muddy.

I am no carpenter. Frankly, I am all thumbs. I was not going to build instruments as Partch did. My approach to tuning could only evolve from the nature of the instrument I had available, which was the synthesizer. I found confirmation for my approach in the writings of ancient Greek musicians such as Ptolemy, who thought of pitches not in terms of the harmonic series (a phenomenon unknown at the time) but in terms of complexes of ratio-defined intervals. And I found that I could explore the new pitch universe I was lost in interval by interval if, for each interval, I found other, more familiar intervals related to it that I had already explored.

I will offer two examples from my subsequent music.[2]

How Miraculous Things Happen (1997)

In terms of pure, ratio-defined tuning, the interval that musicians call a minor third is defined by 6/5, and a major third by 5/4:

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[2] Both pieces of music I discuss are recorded on a compact disc: Custer's Ghost, Monroe Street msm 60104, available from http://www.monroestreet.com.

pitch:	C	Eb	E
ratio:	1/1	6/5	5/4
cents:	0	315.6	386.3

One of the myriad interesting facts of the mathematics of the octave is that, approximately halfway in-between 5/4 and 6/5 lies another, relatively consonant median ratio: 11/9 (347.4 cents). This means, of course, that in-between the major triad (C E G) and minor triad (C Eb G) one can locate a neutral triad with an 11/9 third that is neither major nor minor. (Charles Ives was fascinated with this possibility when he worked with quarter-tones in the 1920s.) In fact, it is the tendency of 11-based intervals to generate what we think of as quarter-tones.

Once given this information, the temptation to write music in which a minor tonality gradually slid into major through that 11/9 interval was irresistible. For maximum illusionary smoothness, I wanted my pitch line to get a running start. I postulated a series of fairly simple ratios through which my long-term melody would rise in the key of A major from B to C#:

pitch:	B-	B	B*L*	C7	C	C↑-	C#
ratio:	10/9	9/8	8/7	7/6	6/5	11/9	5/4
cents:	182.4	203.9	231.2	266.9	315.6	347.4	368.3

I also wanted a pitch line moving in the opposite direction down to C# from D:

pitch:	C#	C#*L*	D7	D-
ratio:	5/4	9/7	21/16	4/3
cents:	386.3	435.1	470.8	498.0

Ultimately, then, I based the music around a succession of ten pitches within what musicians define as a minor third (to be precise, 316 cents), from B to D in the key of A. Again, the accidental notation here is Ben Johnston's.

To simply play this line in isolation is tonally meaningless, and the ear does not automatically relate it to the key of A major. But by supporting each pitch in that line with harmonies made of pitches that relate each pitch to A 1/1, one can create a meaningfully microtonal chord progression that derives a certain musical inexorability through ultrachromatic voice-leading.

For instance, the combination of the tonic A 1/1 and the opening B- at 10/9 implies a third pitch of which both pitches are harmonics. That pitch is a very low G-. Transpose it up into the same octave and you will have G- 16/9. The B- 16/9 is harmonized with the major triad:

B-	10/9	16/9 x 5 = 80/9 80/9 ÷ 8 = 10/9	
D-	4/3	16/9 x 3 = 16/3	16/3 ÷ 4 = 4/3
G-	16/9		

A is the 9th harmonic of G-, and B- is the 10th (or 5th) harmonic. Therefore a triad on G- makes the relationship between A and B- intelligible.

A similar logic finds a similar chord with which to harmonize C↑- 11/9. This pitch is the 11th harmonic of the pitch of which A is the 9th harmonic: thus 11/9. Again, the implied pitch, of which both 1/1 and 11/9 are harmonics, is G- 16/9.

The ten melodic pitches that run through How Miraculous Things Happen are harmonized with the ten chords shown in Example 1. Each chord serves the function of intelligibly relating its melody pitch to the tonic A. The pitch gamut of most of the piece was drawn from the pitches needed for these ten chords. At the end of the work, a few additional pitches appear for the purpose of transposing the 11/9 note to the key of F#.

This may be the best opportunity for explaining Ben Johnston's notation:

+ raises a pitch by 81/80, or 21.5 cents.

- lowers by the same amount.

7 adjusts for the 7th harmonic by lowering a pitch by a ratio of 35/36, or 48.8 cents.

L, an upside-down 7, raises a pitch by 36/35, or 48.8 cents.

↑ raises a pitch to an 11th harmonic by multiplying it by 33/32, or adding 53.3 cents.

↓ lowers by the same amount, 32/33 or 53.3 cents.

has the same value it usually did in 16th century theoretical treatises, 25/24 or 70.7 cents.

b likewise lowers by 24/25, or 70.7 cents.

In addition, it is assumed that C E G, F A C, and G B D are all purely tuned triads in a 4:5:6 ratio.

Note that in the musical examples, some of these accidentals are combined: an arrow attached to a sharp, for example, or a line attached to the top or bottom of a sharp to add a 7 or *L*. Pluses and minuses, however, are never attached to other accidentals.

It might be noted that the pitch C#, the goal pitch of the melodic line, is harmonized not with an A major triad as might be expected, but with F# minor. What keeps happening in the piece is that A minor keeps trying to turn into A major (through that 11/9 interval), but every attempt is thwarted by the bass line moving to F#. At last, in the final measures of the piece, the entire process shifts to the new key, and F# minor transforms into F# major via that same 11/9 (actually 55/54 now, with respect to A: 11/9 x 5/3 = 55/27 or 55/54). The desired goal was blocked, but finally opened up again in a new channel, which - or so it seemed to me at the time - is how miraculous things happen.

"Sun Dance" from <u>Custer and Sitting Bull</u> (1999)

In the "Sun Dance" scene from my vocal/electronics piece <u>Custer and Sitting Bull</u>, the progression became somewhat more complex.

The challenge of the movement was to depict the sun dance that Sitting Bull led prior to the Battle of the Little Bighorn. During the dance, Sitting Bull cut a hundred notches of flesh in his arms and legs, letting the blood run down until he had a vision. The vision was of white soldiers and cavalry falling down, as a voice said, "I give you these because they have no ears."

I wanted the music, then, to express a trancelike state through a very slow resolution of dissonance, dissonance that was dwelt upon and savored at some length. The harmonic skeleton of the movement moves gradually, over and over, through a series of intervals moving from a perfect fourth above the tonic F# (actually a slightly smaller interval to begin with) through several definitions of tritone up to a perfect fifth:

pitch:	B7	B-	B	B↑-	C7	B#*L*	C#↓	C#-	C#
ratio:	21/16	4/3	27/20	11/8	7/5	10/7	16/11	40/27	3/2
cents:	470.8	498.0	519.6	551.3	582.5	617.5	648.7	680.5	702.0

One tritone that is not used is the traditional 45/32 tritone (590.2 cents) of early European tunings, which is a complex, derived interval arrived at by adding a major second (9/8) to a major third (5/4). Only 8 cents away, 7/5 rendered 45/32 superfluous for my purposes, and offered more exotic possibilities.

This "scale" - although the pitches are closer than I would use for a scale in the sense of a series of steps that a melody could move up and down on - represents an intriguing progression of relative consonance and dissonance. 4/3 and 3/2 are among the most consonant possible intervals. Moving away from either of them, one quickly encounters much more dissonant intervals with a tendency to form acoustic beats: 21/16, 27/20, 40/27. Moving still further, one encounters 11-based intervals that are more consonant, but close to quarter-tones within a European-based musical system: 11/8 and 16/11 (both intervals common in Arabic music). And in the middle one finds two strikingly consonant tritones, 7/5 and 10/7, even though European history has never allowed these tunings for the much-maligned (and unfairly so, clearly) "devil's interval."

Each interval is accompanied by a scale which makes the interval pitches more intelligible. Since the derivation of six pitches from 40/27 has already been done, let us take its inverse, 27/20:

$$\frac{27}{20} = \frac{3 \times 3 \times 3}{2 \times 2 \times 5}$$

Taking subsets of these numbers and multiplying freely by two, we obtain:

$$\frac{3 \times 2}{5} = \frac{6}{5} \qquad \frac{3 \times 3}{5} = \frac{9}{5} \qquad \frac{3}{2} \qquad \frac{3^0}{5} = \frac{8}{5}$$

$$\frac{3 \times 3}{2 \times 2 \times 2} = \frac{9}{8} \qquad \frac{3 \times 3 \times 3}{2 \times 2 \times 2 \times 2} = \frac{27}{16}$$

This series of pitches, then:

Ratio:	Pitch name:	Cents above F#:
9/5	E	1017.6...
27/16	D#	905.9...
8/5	D-	813.7...
3/2	C#	702.0...
27/20	B	519.6...
6/5	A	315.6...
9/8	G#	203.9...
1/1	F#	0.0

defines a scale of which each note (besides 1/1 and 27/20) is more consonant with 1/1 and 27/20 than they are with each other; the scale makes the relationship of 27/20 to 1/1 intelligible.

The intervals from 21/16 to 3/2 run through the bass of the "Sun Dance" as drones. Over each drone, scale notes (not always the entire gamut available) are used as in Example 2. At various points in the middle of the movement, quite active beats are audible, caused by a complex relation between the drone pitches. At the point of maximum consonance, when 3/2 is reached, the voice enters (my own voice pitch-shifted downward, actually) chanting "I give you these because they have no ears." The effect is of pitch relations going gradually out of, and then back into, focus over several minutes. A page of the score is given in Ben Johnston's notation in Example 3.

The "Sun Dance" movement concludes with the "Battle of the Greasy-Grass River," which is a translation of the Sioux name for the Little Bighorn. At the end of the battle depiction (which juxtaposes two tonalities a 10/7 apart, one representing the Sioux and the other the U.S. Cavalry), I made my most extreme exploration yet of ratio-defined dissonance. As the bodies lay mutilated on the battlefield, I wanted sustained chords that would suggest the stillness of death, but whose pulsing beats would nevertheless writhe with all the charged emotionality of the scene. I selected, then, the most mutually-complexly-related pitches I had available to compose the chords shown in Example 4.

Superficially, these are generally seventh or ninth chords of various types revolving around the tonic F#, but the extreme incommensurability of the individual notes results in a wild array of sparkling beats, vividly audible at sufficiently high volume. The final chord, the most dissonant, is an interval complex of ratios reducible no smaller than 3780:5103:5600:6750 - which means that it takes those numbers of vibrations for the pitches to all return to their original phase relation. In addition, I added small, steadily rising glissandos to the lower drone notes to subtly change the rate of beating within each chord.

One will note that that final dissonance, like much of the piece, is based on the wolf fifth 40/27. It seemed appropriate that the spirit of the wolf should pervade an American Indian scene, and particularly that the wolf should howl at that tragic moment. The whole experiment confirmed for me what Harry Partch so eloquently said about the difference between equal-tempered dissonance and purely-tuned dissonance: it's "a whole different serving of tapioca."

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My mathematical explanations of my music sometimes create the mistaken impression that my music is highly precompositionally structured, that I write "mathematical music." Actually, I belong to a theoretical tradition that I see as peculiarly American, starting with Henry Cowell's book New Musical Resources. It is common, within this tradition, to expend abundant theoretical energy developing the raw materials with which one composes; then, when one actually comes to compose, to proceed freely and intuitively with few or no precompositional determinants. An analogy would be a painter who goes to tremendous lengths to chemically design his own paints and colors from scratch, but who then, having completed that task, picks up his brush and, as spontaneously as possible, paints what he sees. Any theoretician of the future who finds Golden Sections, Fibonacci series', pitch rows, Schenkerian urlinie, and other global structural devices in my music can rest assured that they were not put there consciously. I am no proponent of mathematical methods of composing (though I am not necessarily opposed to them either); but the materials of which music is made are inherently mathematical, or at least arithmetical. And when it comes to pitch materials, I prefer to create my own pure, organic ones rather than use the presets that come out of that mechanical box, the piano.

Of course, the way I design a tuning imposes its own structure on my music. This is true of all music, a fact rarely recognized. Every piece of pitch-oriented music is an exploration of a tuning; the tuning pervasively influences where the music can go, what sounds good in it, what types of behavior it will follow. I feel more comfortable being conscious of this fact than I used to be when the influence was unconscious. One problem with music at the moment is that thousands of composers have spent the last hundred years exploring one of the dullest, most colorless, least fulfilling tunings ever devised by man.

There are as many approaches to microtonality as there are microtonal composers. My approach aims at making microtonal intervals intelligible. To take a common and representative example, the F that defines the subdominant in a C major scale is not the same F that acts as the 7th harmonic in a dominant 7th chord on G; the first F, 4/3, is 498 cents above C, and the second, 21/16, is 471, more than a fourth of a half-step flatter. My microtonal music recognizes this difference, which the ear intuitively picks up and comfortably assimilates if given half a chance.

The advantage of working the way I do is that I can go further and further out and always find my way back. Let's say I, for whatever whimsical reason, become enamored of the interval 105/88.

$$\frac{105}{88} = \frac{3 \times 5 \times 7}{2 \times 2 \times 2 \times 11}$$

I can make that interval meaningful, intelligible in the piece by approaching it, or coming back from it, or harmonizing it, via the other pitches implied: 35/32, 35/22, 15/11, 15/8, 14/11, 18/11, 7/4 and so on. Hopefully I can bring the listener to feel where that note is in relation to the tonic or some familiar chord, rather than simply introduce it as some strange, arbitrary pitch 305.8 cents away.

My experience with other people's perception of my music has suggested a recurring pattern. At first hearing, the music seems bizarrely out of tune. But on subsequent second and third hearings, the chord progressions quickly acquire a feeling of inevitability, thanks to the purity of the harmonies and the consistency of the microtonal voice leading. I am wildly in favor of the diversity represented by the range of microtonal composers, but I do believe that, if microtones are going to become common practice once again - and the imperialist hegemony of this horridly bland, beat-ridden, 12-pitch equal-tempered scale be broken once and for all - we need a body of music representing a microtonal practice that makes theoretical sense to the ear. My music and this article are an attempt to begin providing that. We need not cling to the infinite web that runs throughout the fluid universe of pitch, but if we want to learn to navigate that universe, we have nothing better to cling to.

Example 1: How Miraculous Things Happen
Harmonic skeleton

Pitches in Ben Johnston's notation:

Ratios to A 1/1:

10/9	9/8	8/7	7/6	6/5	11/9	5/4	9/7	21/16	4/3
16/9	15/8	1/1	1/1	1/1	10/9	1/1	8/7	9/8	1/1
4/3	3/2	12/7	14/9	3/2	16/9	5/3	12/7	15/8	5/3
	21/16	10/7	4/3		4/3			3/2	

Cents above A:

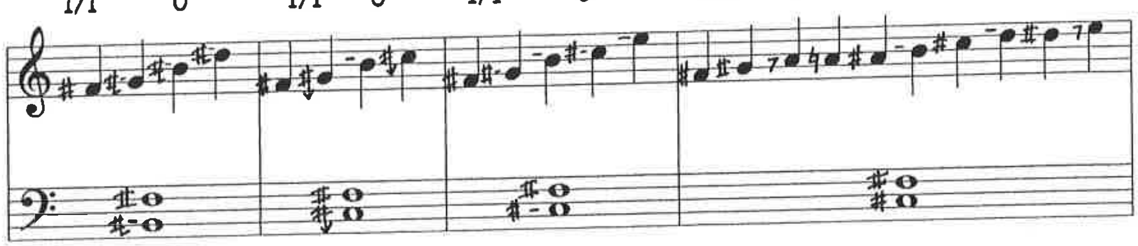
182	204	231	267	316	347	386	435	471	498
996	1088	0	0	0	182	0	231	204	0
498	702	933	765	702	996	884	933	1088	884
	471	617	498		498			702	

Example 2: "Sun Dance" from <u>Custer and Sitting Bull</u>
Drone pitches and related scales

Ratio:	Cents:								
7/4	968.8			9/5	1017.6			9/5	1017.6
3/2	702.0	16/9	996.1	27/20	519.6	3/2	702.0	8/5	813.7
21/16	470.8	40/27	680.5	6/5	315.6	11/8	551.3	7/5	582.5
7/6	266.9	4/3	498.0	9/8	203.9	9/8	203.9	6/5	315.6
1/1	0	1/1	0	1/1	0	1/1	0	1/1	0

Ratio:	Cents:								
21/16	470.8	4/3	498.0	27/20	519.6	11/8	551.3	7/5	582.5

Ratio:	Cents:								
				16/9	996.1	7/4	968.8	5/4	386.3
12/7	993.1	16/11	648.7	40/27	680.5	5/3	884.4	6/5	315.6
10/7	617.5	4/3	498.0	4/3	498.0	8/5	813.7	7/6	266.9
8/7	231.2	12/11	150.6	10/9	182.4	3/2	702.0	9/8	203.9
1/1	0	1/1	0	1/1	0	4/3	0	1/1	0

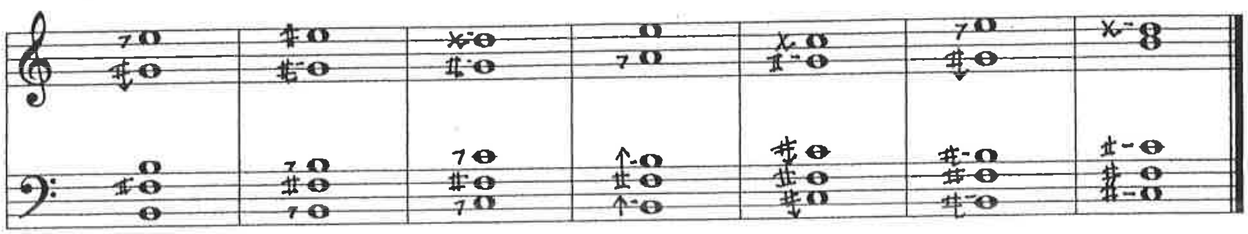


Ratio:	Cents:						
10/7	617.5	16/11	648.7	40/27	680.5	3/2	702.0

Example 3: Measures 48-51 of "Sun Dance/Battle of the Greasy-Grass River" from Custer and Sitting Bull (in Ben Johnston's pitch notation)

Example 4: "Sun Dance/Battle of the Greasy-Grass River
from Custer and Sitting Bull
Dissonant chords following the Battle Scene

(Actual chords an octave lower)



Ratios to F# 1/1:

7/4	15/8	25/14	9/5	45/28	7/4	25/14
12/11	8/7	8/7	7/6	10/9	12/11	27/20
1/1	1/1	1/1	1/1	1/1	1/1	1/1
27/20	21/16	7/5	11/8	16/11	10/7	40/27

Cents above F#:

968.8	1088.3	1003.8	1017.6	821.4	968.8	1003.8
150.6	231.2	231.2	266.9	182.4	150.6	519.6
0	0	0	0	0	0	0
519.6	470.8	582.5	551.3	648.7	617.5	680.5